3. Physical Symmetries

3.1. States and properties in Quantum Φ-sics

Quantum Physics

dense subspace countable (ex: L^2)

properties in Q.Φ. are described by closed sub-vector spaces of a (countable, "complex") Hilbert space H.

see physic as an experiment

ex: 

each S_i is a slot, electron can pass by each and we 0/1 if electron pass in a slot

=> projection of one of the possible outcome.

Equivalently, a property is described by corresponding (orthogonal) projection P.V.

a projection is defined by: $p_V^2 = p_V$

orthogonal projection: the projection is orthogonal to the image V: $p_V^* = p_V$

self-adjoint

ex: $p = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ and $p' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

p is not $\perp$ projection, p' is $\perp$ projection, p' is self-adjoint

A pure state is described by one-dimensional subspace of Hilbert space H. These are the rays of H, denoted by $[v]$, $v \in H \setminus \{0\}$

maximum knowledge of our state

The Hilbert space comes with a norm and scalar product => Topology

The set of all these rays will be denoted by $\mathbb{P}(H)$, the projectivisation of H

There is a canonical projection $\Pi : H^* \longrightarrow \mathbb{P}(H)$, $v \longrightarrow [v]$

We choose $\mathbb{P}(H)$ with the quotient topology, $\tau_\Pi := \{ \mathcal{U} \subset \mathbb{P}(H) \mid \Pi^{-1}\{\mathcal{U}\} \in \tau_H \}$

exercise: show that τ_H is a topology on $\mathbb{P}(H)$ and its the largest topology on $\mathbb{P}(H)$ s.t. Π is continuous

Under τ_H, Π is an open map

For given two states $[v], [w] \in \mathbb{P}(H)$, prob. is given by $\mathrm{Tr}(P_v P_w)$ ↳ transition probability

in Dirac notation $\frac{|v\rangle\langle v|}{\|v\|^2}$ and $\frac{|w\rangle\langle w|}{\|w\|^2}$ $\Rightarrow \mathrm{Tr}\left(\frac{|v\rangle\langle v|}{\|v\|^2}\frac{|w\rangle\langle w|}{\|w\|^2}\right) = \frac{|\langle v|w\rangle|^2}{\|v\|\|w\|^2}$

For $[x],[y] \in \mathbb{P}(H)$, we shall write $[x] \perp [y]$ if $x \perp y$, its equivalent to $P_x P_y = 0$.

If $B \subset \mathbb{P}(H)$, we shall write $B^\perp := \{[y] \in \mathbb{P}(H) \mid \forall [x] \in B,\ [x] \perp [y]\}$

As an exercice, $\overline{B} = \Pi\left(\overline{\Pi^{-1}\{B\}}\right)$ and $\overline{B} = (B^\perp)^\perp$

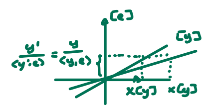
closed subspace on H because $(^\perp)^\perp = \overline{}$

Let $e \in H$ be a unique vector. Consider $V := e^\perp = \overline{V} \subset H$. Hence $U_e := \mathbb{P}(H) \setminus \mathbb{P}(V)$ is open for τ_Π

In fact, $U_e = \{[y] \in \mathbb{P}(H) \mid [y] \in \mathbb{P}(e)\}$

V_e is therefore an open neighbourhood of $[e]$ in $\mathbb{P}(H)$

for some $[y] \in U_e$ consider $X([y]) = \frac{y - (y,e)e}{(y,e)}$



↳ allow to give coordinates

$x_e : U_e \to e^\perp$ with $x_e^{-1}(v) = [e+v]$

↳ this map is a homeomorphism between U_e and $e^\perp$ ↳ bijective + continuous inverse + continuous

↳ check as an exercice

so $\mathbb{P}(H)$ is a Hilbert-manifold with atlas $\{(U_e, x_e) \mid e \in S(H)\}$

3.2. Symmetries in QP

Def: A morphism τ is a map $\tau : \mathbb{P}(H) \to \mathbb{P}(H)$ continuous with τ_Π

A symmetry S is an isomorphism $S : \mathbb{P}(H) \to \mathbb{P}(H)$ so that, if $[x],[y] \in \mathbb{P}(H)$ implies that $\mathrm{tr}(P_x P_y) = \mathrm{tr}(P_{x'} P_{y'})$ $\left|\begin{array}{l}[x'] = S([x]) \\ [y'] = S([y])\end{array}\right.$

↳ conserved transition probabilities ↳ reversible continuous map

Lemma: Let $\tau : \mathbb{P}(H) \to \mathbb{P}(H)$ be a symmetry, let $B \subset \mathbb{P}(H)$, then $\tau(B^\perp) = \tau(B)^\perp$

↳ proof : in exercise $\square$

Corollary: For an isomorphism $\tau : \mathbb{P}(H) \to \mathbb{P}(H)$

let $B \subset \mathbb{P}(H)$, $\tau(\overline{B}) = \overline{\Pi(B)}$

Recall: we shall write $\mathcal{L}(H)$ for the set of bounded linear operators define on H.

| **unitary operator** satisfy $U^* U = 1_H = U U^*$ (preserved the norm => bounded)

| **isometric operators** satisfy $U^* U = 1_H$

Wigner's theorem: Let $e_1, e_2 \in H$ be two orthonormal vectors. Let S be a physical symmetry so that $S[e_1] = [e_2]$ then there exists a unique unitary operator $U \in \mathcal{L}(H)$ such that:

| $U e_1 = e_2$

| $\Pi \cdot U = S \cdot \Pi$